Subject 21H_Q2　　　　Date 0

Attendants

① Triangle inequality　(劣加法性)
subadditivity

$$\|v+u\| \leq \|v\| + \|u\|$$

② absolute homogeneity　(斉次性)

$$\|a u\| = |a|\|u\|$$

③ positive definiteness　$\|u\| = 0 \iff u = 0$

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Attendants

$$x_{t+1} = x_t - \eta(Ax_t + b) = F(x_t)$$

$\boxed{1.}$ $x \in \mathbb{R}^d$

$x \mapsto \bar{x}^T x$ is norm on $\mathbb{C}^d$

$\bar{x} = a - ib$

$x = a + ib$ where $a, b \in \mathbb{R}^d$

So

$$\bar{x}^T x = \sum_{\ell=1}^{d} (a_\ell - ib_\ell)(a_\ell + ib_\ell) \in \mathbb{R}$$

$$= \sum_{\ell=1}^{d} a_\ell^2 + b_\ell^2$$

this is norm on $\mathbb{C}^d$,

$x \mapsto x^T x$ is norm on $\mathbb{R}^d$

$x = a + ib$ where $b = 0$

so $x^T x = \sum_{\ell=1}^{d} a_\ell a_\ell$

$$= \sum_{\ell=1}^{d} (a_\ell)^2 \in \mathbb{R}$$

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A is potentially non-symmetric.

but $x^T A x > 0$

$A \succ 0$ positive definite

$\boxed{2}$ Show that $\Re(\lambda) \geq 0$

$\forall \lambda \in \underbrace{Sp}_{\text{spectrum}}(A)$

we arrange $Sp(A)$ as follows.

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda d$$

since A is positive definite

$\lambda d > 0$ so.

$$\underline{\Re(\lambda) > 0}$$

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Attendants

$\boxed{3}$ Show that there is exists a unique $x^* \in \mathbb{R}^d$ such that $x^* = F(x^*)$

Give its expression as a func of A and b

$$x^* = F(x^*)$$
$$\Leftrightarrow \quad x^* = x^* - \eta(Ax^* + b)$$
$$\Leftrightarrow \quad 0 = Ax^* + b$$
$$\Leftrightarrow \quad x^* = -A^{-1}b$$

since $\lambda_{min}(A) > 0$ so A^{-1} exist.

$\boxed{4}$ $\forall t \geq 0$, there exists a polynomial $P_t(X)$ s.t.

$$x_{t+1} - x^* = P_t(A)(x_0 - x_*)$$

$$x_{t+1} - x^* = x_t - \eta(Ax_t + b) - x^*$$
$$= x_t + \eta\{A(x_{t-1} - \eta(Ax_{t-1} + b)) + b\} - x^*$$

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$$x_{t+1} - x^* = x_t - \eta(Ax_t + b) - x^*$$

$$= x_t + \eta\{A\underbrace{(x_{t-1} - \eta(Ax_{t-1} + b))}_{x_t} + b\} - x^*$$

$$= x_t + \eta\{A(x_{t-1} - \eta[A(x_{t-2} - \eta(Ax_{t-2} + b)) + b]) + b\} - x^*$$

$$= x_t + \eta\{Ax_{t-1} + b - \eta(Ax_{t-1} + b)\} - x^*$$

$$= x_t + \eta\{(I - \eta A)x_{t-1} + b(1-\eta)\} - x^*$$

$$= x^* + \eta\{(I - \eta A)(x_{t-2} - \eta(Ax_{t-2} + b)) + b(1-\eta)\} - x^*$$

$$= x^* + \eta\{(I-\eta A)(I - \eta A)x_{t-2} - \eta(I - \eta A)b + b(1-\eta)\}$$

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$$= x^t + \eta \left\{ (I-\eta A)^2 x_{t-2} - \eta (I-\eta A) b + b(1-\eta) \right\} - x^*$$

$$= x^t + \eta \left((I-\eta A)^2 \underbrace{\left(x_{t-3} - \eta (A x_{t-3} + b) \right)}_{(I-\eta A) x_{t-3} - \eta b} + \cdots \right\} - x^*$$

$$= x^t + \eta \left\{ (I-\eta A)^3 x_{t-3} - \eta b (I-\eta A)^2 - \eta b (I-\eta A)^1 - \eta b (I-\eta A)^0 + b \right\} - x^*$$

$$= x^t + \eta \left\{ (I-\eta A)^t x_0 - \eta b \sum_{k=0}^{t-1} (I-\eta A)^k + b \right\} - x^*$$

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$$x_{t+1} = x_t - \eta(Ax_t + b)$$

$$= (I - \eta A)x_t - \eta b$$

$$= (I-\eta A)\{x_{t-1} - \eta(Ax_{t-1} + b)\} - \eta b$$

$$= (I-\eta A)\{(I-\eta A)x_{t-1} - \eta b\} - \eta b$$

$$= (I-\eta A)^2 x_{t-1} - \eta b(I-\eta A) - \eta b$$

then

$$x_{t+1} = \underbrace{(I-\eta A)}_{\hookrightarrow D}{}^{t+1} x_0 - \eta b \sum_{l=0}^{t} \underbrace{(I-\eta A)}_{\hookrightarrow D}{}^{l}$$

$$x_{t+1} - x^* = D^{t+1}x_0 - \eta b \sum_{l=0}^{t} D^l - x^*$$

from $\boxed{3}$ $x^* = -A^{-1}b$

$\Leftrightarrow b = -Ax^*$

$$= D^{t+1}x_0 + \eta A x^* \sum_{l=0}^{t} D^l - x^*$$

$$= D^{t+1}x_0 - x^*\left(I - \eta A \sum_{l=0}^{t} D^l\right)$$

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$$x_{t+1} - x^* = D^{t+1} x_0 - x^* \left(I - \eta A \sum_{l=0}^{t} D^l\right)$$

$$= (I - \eta A) D^t x_0 - x^* (I - \eta A) \sum_{l=0}^{t} D^l \quad X$$

$$= D^{t+1} x_0 - \sum_{l=1}^{t+1} D^l \, x^* \quad ?$$

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Attendants

$$x_{t+1} = x_t - \eta(Ax_t + b) \qquad x^* = -A^{-1}b$$

$$\Leftrightarrow b = -Ax^*$$

$$= (I - \eta A)x_t - \eta b$$

$$\Leftrightarrow x_{t+1} + \eta b = (I - \eta A) x_t$$

$$= (I - \eta A)\{x_{t-1} - \eta A x_{t-1} - \eta b\}$$

$$= (I - \eta A)\{(I - \eta A) x_{t-1} - \eta b\}$$

$$\Leftrightarrow x_{t+1} = (I - \eta A)^{t+1} x_0 - \eta b \sum_{l=0}^{t} (I - \eta A)^l$$

$$\Leftrightarrow x_{t+1} - x^*$$

$$= (I - \eta A)^{t+1} x_0 + \eta A x^* \sum_{l=0}^{t} (I - \eta A)^l - x^*$$

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Attendants

$$x_{t+1} = x_t - \eta(Ax_t + b)$$

$$= (I - \eta A)x_t - \eta b$$

$$\Leftrightarrow \quad x_{t+1} = (I - \eta A)x_t - \eta b$$

$$b = -Ax^*$$

$$x_{t+1} - \eta A x^* = (I - \eta A)(x_{t-1} - \eta A x_{t-1} + \eta A x^*)$$

$$\Leftrightarrow \quad x_{t+1} = (I - \eta A)\left\{(I - \eta A)x_{t-1} + \eta A x^*\right\} + \eta A x^*$$

$$= (I - \eta A)^2 x_{t-1} + \eta A x^* (1 - \eta A)$$

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Attendants

$$\cancel{w^{t+1} = (1-\alpha\lambda) w^{t} - \alpha(U^{T} b)}$$

$$\cancel{w^{t-1} - w^{*} = (1-\alpha\lambda)^{t} (w^{t} - w^{*})}$$

$$x_{t+1} = x_t - \eta A x_t - \eta b$$

$$b = -Ax^{*}$$

$$\boxed{\begin{aligned} x_{t-1} - x^{*} &= x_t - x^{*} - \eta A x_t + \eta A x^{*} \\ &= (x_t - x^{*}) - \eta A (x_t - x^{*}) \\ &= (I - \eta A)(x_t - x^{*}) \end{aligned}}$$

$$\underline{x_{t-1} - x^{t} = (I - \eta A)^{t} (x_t - x_0)}$$

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Attendants $\boxed{6}$

$\rho(B) = \lambda_{max}(B)$ is a norm on $\mathbb{R}^{d \times d}$?

$\longrightarrow \lambda_{max}(B) = \lambda_{max}(\Lambda)$

where $B = U \Lambda U^T$

$\rho(B) : \mathbb{R}^{d \times d} \longmapsto \mathbb{R}^d$

$\Downarrow$

$\mathbb{R}$

Yes

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Attendants

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$\boxed{7}$ $|||P_t(A)||| < r^t$ $\quad \eta > 0$

$$(I-\eta A)^t < r^t$$

$$(x_{t+1} - x^*) = (I-\eta A)^t (x_0 - x^*)$$

$\Longleftrightarrow$

$$\frac{(x_{t+1} - x^*)}{(x_0 - x^*)} = (I-\eta A)^t < r^t$$

$$\Longleftrightarrow \quad \frac{(x_{t+1} - x^*)}{(x_t - x^*)} = (I-\eta A) < r$$

it is linear convergence //

$$O((I-\eta A)^t)$$ //