Subject PreDoc2 A21 Part II

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March 31st, 2022

Q2 Circular convolution is the only linear operation with shift equivarance

$$f(Sx) = Sf(x)$$

$f(Sx) \Rightarrow Sf(x)$

function f which holds above equation

Thanks to Ioannis.

Operator A : that is linear $\left(\begin{array}{l} A(X+Y) = AX + AY \cdots ① \\ A(cX) = cAX \cdots ② \end{array} \right.$

with shift equivarance $(SA = AS) \cdots ③$

where S is circular right shift operator

Goal : Prove $A = C_a$ (circulants)

Proof :

From ③ $(AS)_{ij} = (SA)_{ij} \Rightarrow e_i^T A S e_j = e_i^T S A e_j$

where $e_i = (0 \ldots 1 \cdots 0)$

i-th element

⟨Left Term⟩

$$e_i^T A S e_j = e_i^T A (S e_j)$$

$$S = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

S_0

Se_j is as follows

$$S \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = S_j$$

j-th row

$$S = (S_0^T, S_1^T \cdots S_{n-1}^T)$$

$$S_0 = (0, 1 \cdots 0)$$

$$\vdots$$

$$S_{n-2} = (0 \cdots 1)$$

$$S_{n-1} = (1 \cdots 0)$$

A_0

$$A = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0n-1} \\ a_{10} & a_{11} & \cdots & a_{1,n-1} \\ \vdots & & & \\ a_{n-1,0} & \cdots & & a_{n-1}a_{n-1} \end{pmatrix}$$

$$e_i^T A = (0, \cdots 1 \cdots 0) A = \underline{\underline{A_i}}$$

i-th column

$$A = (A_0, A_1, \cdots A_{n-1})^T$$

Then

$$e_i^T A S e_j = A_i S_j = (a_{i0}\ a_{i1} \cdots a_{in-1}) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

← j+1 mod n th row

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Finally

$$A_i \delta_j = a_{i,(j+1) \bmod n}$$

Seems to be correct

<Right Term>

$$e_i^T S A e_j = (e_i^T S)(A e_j)$$

$$S = (\delta_0, \delta_1 \cdots \delta_{n-1})^T$$

$$\begin{array}{c} \\ (0 \cdots 1 \cdots 0) \\ \uparrow \\ i\text{-th column} \end{array} \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots 1 & 0 \end{pmatrix} = \delta_i$$

δ_0

$$\delta_0 = (0 \cdots 1)$$

$$\vdots$$

$$\delta_{n-1} = (0 \cdots 1 \, 0)$$

$(i-1) \bmod n$ -th column

$$A e_j = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0\,n-1} \\ a_{10} & a_{11} & \cdots & a_{1\,n-1} \\ \vdots & & & \\ a_{n-1\,0} & \cdots & \cdots a_{n-1} & a_{n-1} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = A_j$$

A_0

Then

$(i-1) \bmod n$ -th column

$$e_i^T S A e_j = \delta_i A_j = (0 \cdots 1 \cdots 0) \begin{pmatrix} a_{0j} \\ a_{1j} \\ \vdots \\ a_{n-1\,j} \end{pmatrix}$$

$$= a_{(i-1) \bmod n,\, j}$$

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$$\underbrace{e_i^T A S e_j}_{\downarrow} = \underbrace{e_i^T S A e_j}_{\downarrow}$$

Operator A has to hold this property

$$a_{i,j+1 \bmod n} \qquad a_{(i-1) \bmod n, j}$$

$i=1, j=0$ $\boxed{a_{11}}$ $\boxed{a_{00}}$

$i=1, j=1$ a_{12} a_{01}

$$C_a = \begin{bmatrix} \boxed{a_0} & a_{n-1} & a_{n-2} & \cdots & a_1 \\ a_1 & \boxed{a_0} & a_{n-1} & \cdots & a_2 \\ a_2 & a_1 & a_0 & & \vdots \\ \vdots & \vdots & a_1 & & \vdots \\ & & \vdots & & \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{bmatrix}$$

which is circular convolution operator

$A = C_a$

$$= \begin{bmatrix} \boxed{a_{00}} & a_{01} & a_{02} & \cdots & a_{0\,n-1} \\ a_{10} & \boxed{a_{11}} & a_{12} & \cdots & a_{1\,n-1} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2\,n-1} \\ \vdots & \vdots & a_{23} & \ddots & \vdots \\ & & \vdots & & \\ a_{n-1,0} & \cdots & \cdots & a_{n-1} & a_{n-1} \end{bmatrix}$$