17A Q3-9

Asymptotic Analysis

eq3 $\quad Q(w_*) = \left\langle \frac{w_*}{\|w_*\|^2}, v_1 \right\rangle^2 \in [0,1]$

Quality of estimate w_*

$$Q(z_*) = 1 - \underbrace{\frac{\sum_{i=2}^{n} \lambda_i^{2*} z_0^2(i)}{\sum_{i=1}^{n} \lambda_i^{2*} z_0^2(i)}}_{\text{approximation error}}$$

$$Q(z_*) = \frac{\sum_{i=1}^{n} \lambda_i^{2*} z_0^2(i)}{\sum_{i=1}^{n} \lambda_i^{2*} z_0^2(i)} - \frac{\sum_{i=2}^{n} \lambda_i^{2*} z_0^2(i)}{\sum_{i=1}^{n} \lambda_i^{2*} z_0^2(i)}$$

$$Q(z_t) = \frac{\lambda_1^{2t} z_0^2(1)}{\sum_{i=1}^{n} \lambda_i^{2t} z_0^2(i)}$$ is what I want to show.

p-norm

$$(|x_1|^p + |x_2|^p \cdots |x_n|^p)^{\frac{1}{p}}$$

$$Q(w_t) = \left\langle \underbrace{\frac{w_t}{|w_t|_2}}, v_1 \right\rangle^2$$

$$w_t = \Sigma w_{t-1}$$

$$= \frac{\langle w_t \cdot v_1 \rangle^2}{|w_t|_2^2} = \frac{\langle \Sigma^t w_0\, v_1 \rangle^2}{|w_t|_2^2}$$

$\Sigma v_i = \lambda_i v_i$
from Def of Eigenvalue.

$$= \frac{\langle \quad \rangle^2}{|w_t|_2^2}$$

$$z_t(i) = \lambda_i^t \underbrace{z_0(i)}$$

$$= \frac{\langle \Sigma^t v_1, w_0 \rangle^2}{\cdots}$$

$$z_0(i) = \langle v_i\, w_0 \rangle$$

$$= \frac{\langle \lambda_i^t v_1, w_0 \rangle^2}{|w_t|_2^2}$$

$$\frac{\lambda_i^{2t} z_0^2(1)}{|w_t|_2^2}$$

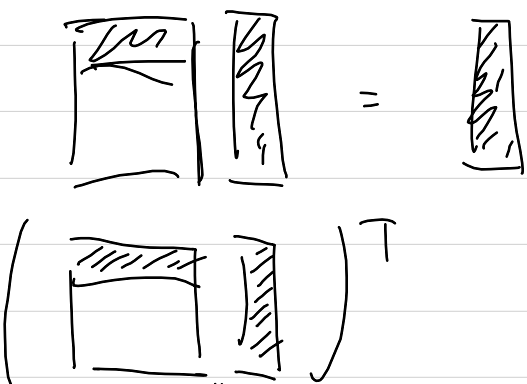
$$Q(w_t) = \frac{\lambda_i^{2t} z_0^2(i)}{|w_t|_2^2}$$

$$\Sigma v_i = \lambda_i v_i$$

$$w_t = \Sigma w_{t-1} = \Sigma^t w_0$$

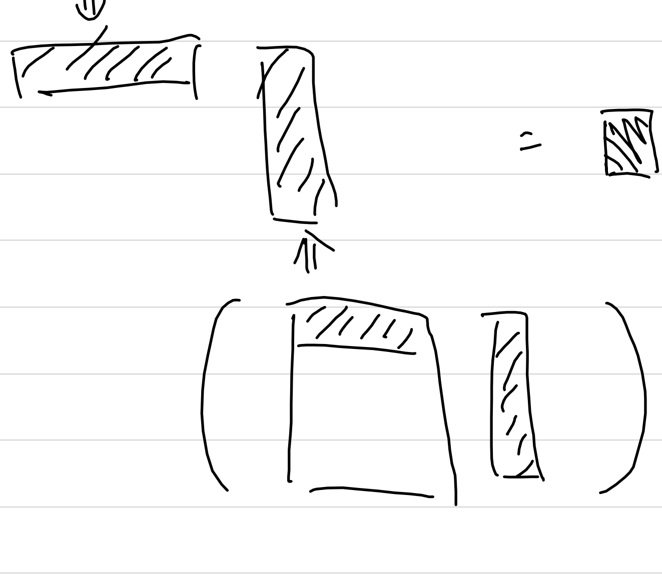
$$|w_t|_2^2 = \langle \Sigma^t w_0, \Sigma^t w_0 \rangle$$

sum$(\lambda_i^{2t} z_0^2(i))$



$z_0(i) = v_i w_0$

$\Sigma v_i = \lambda_i v_i$



from what we want to show

$\hookrightarrow \|w_t\|_2^2 = \sum_{i=1}^n \lambda_i^{2t} \gamma_0^2(i)$

$= \sum_{i=1}^n w_{ti}^2$ $\qquad \langle v_i, w_0 \rangle$

$\sum_{i=1}^n \left(\lambda_i^t v_i w_0\right)^2$

$$w_t = \begin{bmatrix} w_{t1} \\ \vdots \\ w_{tn} \end{bmatrix} = \begin{bmatrix} \Sigma \end{bmatrix}^t \begin{bmatrix} w_0 \end{bmatrix}$$

$\Sigma v_i = \lambda_i v_i$

$\Sigma^{2t} v_i = \lambda_i^{2t} v_i$

Ah! $\|v_i\|_2^2 = 1$

let's use above one.

$\Sigma v_i = \lambda_i$

$|W_t|_2^2 = |w_t|_2^2 \, \underbrace{|v_i|_2^2}$

Cauchy-Shwarz inequality

eigenvector is unit-length vector

$= |\Sigma^t w_0|_2^2 \, |v_i|_2^2$

should be $\geq$

$= |\Sigma^t v_i w_0|^2$

$\Sigma^t v_i = \lambda_i^t v_i$

$= |\sum_{i=1}^n \lambda_i^t \underbrace{v_i w_0}|^2$

$= \sum_{i=1}^n \lambda_i^{2t} \, z_0^2(i)$ ... $\otimes$

So, $Q(w_t) = \left\langle \frac{w_t}{|w_t|_2}, v_1 \right\rangle^2$

$= \frac{1}{|w_t|_2} \times \langle W_t, v_1 \rangle^2$

$= \frac{1}{|w_t|_2} \langle \Sigma^t w_0, v_1 \rangle^2$

$\Sigma v_1 = \lambda_1 v_1$

$= \frac{1}{|w_t|_2} \langle \Sigma^t v_1, w_0 \rangle^2$

$= \frac{1}{|w_t|_2} \lambda_1^{2t} \langle v_1, w_0 \rangle^2 = \frac{1}{|w_t|_2} \lambda_1^{2t} z_0^2(1)$

$$Q(w_t) = \frac{1}{|w_t|_2}\left(\lambda_1^{2t} z_0^2(1)\right)$$

since ⊛

$$= \frac{\lambda_1^{2t} z_0^2(1)}{\sum_{i=1}^{n} \lambda_i^{2t} z_0^2(i)}$$

$$= \frac{\sum_{i=1}^{n} \lambda_i^{2t} z_0^2(i) - \sum_{i=2}^{n} \lambda_i^{2t} z_0^2(i)}{\sum_{i=1}^{n} \lambda_i^{2t} z_0^2(i)}$$

$$= 1 - \frac{\sum_{i=2}^{n} \lambda_i^{2t} z_0^2(i)}{\sum_{i=1}^{n} \lambda_i^{2t} z_0^2(i)}$$

Q.E.D.

Then, evaluate approximation error

$$\frac{\sum_{i=2}^{n} \lambda_i^{2t} z_0^2(i)}{\sum_{i=1}^{n} \lambda_i^{2t} z_0^2(i)} = \frac{\sum_{i=2}^{n} \lambda_i^{2t} z_0^2(i)}{\lambda_1^{2t} z_0(1) + \sum_{i=2}^{n} \lambda_i^{2t} z_0^2(i)}$$

$$= \frac{\sum_{i=2}^{n} \left(\frac{\lambda_i}{\lambda_1}\right)^{2t} z_0^2(i)}{z_0^2(1) + \sum_{i=2}^{n} \left(\frac{\lambda_i}{\lambda_1}\right)^{2t} z_0^2(i)}$$

$$\leq \frac{1}{z_0^2(1)} \left(\sum_{i=2}^{n} \left(\frac{\lambda_i}{\lambda_1}\right)^{2t} z_0^2(1) \right)$$

$$= O\left(\left(\frac{\lambda_2}{\lambda_1}\right)^{2t} \right)$$