

17A-Q3-f 固有ベクトルと直交行列

Let $U = [u_1 \dots u_p]$... denote

orthogonal matrix of eigenvectors
of Σ

$$\left(\begin{array}{l} \Sigma = U \Gamma V^T \text{ (Singular Value Decomposition)} \\ A = X \Lambda X^{-1} \text{ (Eigenvalue Decomposition)} \end{array} \right)$$

$$U \Sigma U^{-1} = \begin{pmatrix} \lambda_1 & \dots & \lambda_n \end{pmatrix}$$

scalar dynamics,

$$w_{t+1} = \Sigma w_t \quad z_t(i) = \lambda_i^* z_0(i)$$

$$z_0(i) = \langle u_i, w_0 \rangle \quad \leftarrow \text{where}$$

Σ の固有ベクトル u_i と 縦ベクトル w_0 との内積は

行列 P とすると $P^{-1} \Sigma P$ は 対角値を

対角成分に持つ対角行列と変換

縦ベクトル

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \underbrace{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}_{\text{縦ベクトル (column vector)}} = \begin{pmatrix} a_{11}b_1 + a_{12}b_2 \\ a_{21}b_1 + a_{22}b_2 \end{pmatrix}$$

縦ベクトル (column vector)

横ベクトル

$$(a_1 \ a_2) \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = (a_1b_{11} + a_2b_{21}, \ a_1b_{12} + a_2b_{22})$$

$$U = [u_1, \dots, u_n]$$

$$U^{-1} \Sigma U = \Lambda \iff \Sigma U = \Lambda U$$

$$\Sigma u_i = \lambda_i u_i$$

$$\underbrace{\begin{array}{|c|} \hline \begin{array}{ccc} \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \end{array} \\ \hline \Sigma \end{array}}_{\Sigma} \underbrace{\begin{array}{|c|} \hline \begin{array}{c} \square \\ \vdots \\ \square \end{array} \\ \hline u_1 \end{array}}_{u_1} \underbrace{\begin{array}{|c|} \hline \begin{array}{c} \square \\ \vdots \\ \square \end{array} \\ \hline u_n \end{array}}_{u_n} = \underbrace{\begin{array}{|c|} \hline \begin{array}{ccc} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{array} \\ \hline \end{array}}_{\Lambda} \underbrace{\begin{array}{|c|} \hline \begin{array}{c} \square \\ \vdots \\ \square \end{array} \\ \hline u_1 \end{array}}_{u_1} \underbrace{\begin{array}{|c|} \hline \begin{array}{c} \square \\ \vdots \\ \square \end{array} \\ \hline u_n \end{array}}_{u_n}$$

$$\begin{pmatrix} \textcircled{a} & \textcircled{b} \\ \textcircled{c} & \textcircled{d} \end{pmatrix} \begin{pmatrix} \textcircled{x} & \textcircled{y} \\ \textcircled{s} & \textcircled{w} \end{pmatrix} = \begin{pmatrix} \textcircled{1} \textcircled{ax+bs} & ay+bw \\ \textcircled{2} \textcircled{cx+ds} & cy+dw \end{pmatrix}$$

$$\Sigma V = \Lambda V$$

$$\begin{bmatrix} \# & \# & \# \\ \# & \# & \# \\ \# & \# & \# \end{bmatrix} \begin{bmatrix} ||| \\ ||| \\ ||| \end{bmatrix} = \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} \begin{bmatrix} ||| \\ ||| \\ ||| \end{bmatrix}$$

正交ベクトル

これに考えよう

$$V = [u_1 \dots u_n]$$

$$u_i = \begin{pmatrix} \circ \\ \circ \\ \circ \\ \circ \end{pmatrix}$$

↑
に相対する
対角成分

$$\Sigma u_i = \Lambda u_i$$

$$\begin{bmatrix} \# & \# & \# \\ \# & \# & \# \\ \# & \# & \# \end{bmatrix} \begin{bmatrix} ||| \\ ||| \\ ||| \end{bmatrix} = \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix} \begin{bmatrix} ||| \\ ||| \\ ||| \end{bmatrix} \Rightarrow \begin{bmatrix} ||| \\ ||| \\ ||| \end{bmatrix}$$

$$=$$

$\Rightarrow$

$$\begin{bmatrix} ||| \\ ||| \\ ||| \end{bmatrix}$$

このベクトルを x とする

i-th eigenvector

$$\langle u_i, u_0 \rangle = \lambda_0(i)$$

$$x^{(0)} = c_1 \underline{u_1} + \dots + c_n \underline{u_n} \text{ である}$$

基底形独立な固有ベクトル

$$= U c$$

$$x^{(0)} \text{ に } \Sigma \text{ をかけると } x^{(1)} \text{ である}$$

$$\begin{aligned} x^{(1)} &= x^{(0)} = \Sigma U c = \underline{U U^T \Sigma} U c \\ &= U \begin{pmatrix} \lambda_1 & \dots & \lambda_n \end{pmatrix} c \\ &= \lambda_1 c_1 u_1 + \dots + \lambda_n c_n u_n \end{aligned}$$

これを Σ をかけると $x^{(2)}$ である

$$\begin{aligned} x^{(k)} &= \underbrace{\Sigma \dots \Sigma}_{k \text{ 回}} U c = \lambda_1^k c_1 u_1 + \dots + \lambda_n^k c_n u_n \\ &= \lambda_1^k c_1 \left\{ u_1 + \dots + \underbrace{\left(\frac{\lambda_n}{\lambda_1} \right)^k \left(\frac{c_n}{c_1} \right) u_n}_{\downarrow 0} \right\} \end{aligned}$$

$$\lim_{k \rightarrow \infty} x^{(k)} = \lambda_1^k c_1 u_1 = u_1$$

↑ 定数倍の固有ベクトル

$V = [u_1, \dots, u_n]$ 直交ベクトルで Σ の固有ベクトル

$$U \Sigma U^{-1} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \Leftrightarrow U \Sigma = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U$$

$$\begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \Sigma \\ \hline \end{array} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$$

$\uparrow$
 u_1

$$\begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \Sigma \\ \hline \end{array} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$$

$$z_0(i) = \langle u_i, w_0 \rangle$$

$$w_{t+1} = \Sigma w_t$$

行列 Σ が対角化可能とすると
基底独立なベクトル u_i を用いて w_0 を表すベクトル c

$$x_0 = c_1 u_1 + \dots + c_n u_n = U c$$

17より c は z_0 と等しい

$$x_1 = \Sigma x_0 = \Sigma U c = U \Sigma U^{-1} U c$$

$\Leftarrow$

$$= U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} c$$

$$x_{t+1} = \Sigma x_t$$

$$= \lambda_1^t c_1 u_1 + \dots + \lambda_n^t c_n u_n \text{ と表せる}$$

$$= \lambda_1^t c_1 u_1 + \dots + \lambda_n^t c_n u_n$$

Σ は n 個の固有ベクトル $x_1, \dots, x_n$ を持つ

$$w_{k+1} = \sum w_k \lambda_i^{(k+1)} c_i \left\{ x_1 + \left(\frac{\lambda_2}{\lambda_1} \right)^{(k+1)} \frac{c_2}{c_1} x_2 + \dots + \left(\frac{\lambda_n}{\lambda_1} \right)^{(k+1)} \frac{c_n}{c_1} x_n \right\}$$

$\downarrow$
 0

$k \rightarrow \infty$

$$w_{k+1} = \lambda_1^{(k+1)} c_1 x_1 \rightarrow \text{定数倍の固有ベクトル}$$

$$\left(\begin{array}{l} \text{e.g.} \quad x_2 = \Sigma x_1 = U U^{-1} \Sigma x_1 \\ \quad \quad \quad = U U^{-1} \Sigma U \begin{pmatrix} \lambda_1 & \dots & \lambda_n \end{pmatrix} c \\ \quad \quad \quad = U \begin{pmatrix} \lambda_1 & \dots & \lambda_n \end{pmatrix}^2 c \end{array} \right)$$

17A-Q3-f Answer

Let $w \in \mathbb{R}^p$ as an arbitrary vector

w_0 can be expressed by using

sequence of scalar c and

independent vectors $U = [u_1 \dots u_n]$

as follows

$$w_0 = c_1 u_1 + \dots + c_n u_n = \underline{Uc}$$

$$w_1 = \Sigma w_0 = \Sigma \underline{Uc}$$

$$\left(\begin{array}{l} \text{then } U \text{ is eigenvectors of } \Sigma \\ \text{in other word } \Sigma U = U \lambda I \\ \Leftrightarrow U^T \Sigma U = \lambda I \end{array} \right)$$

$$\begin{aligned} w_1 &= \Sigma U c = U U^T \Sigma U c \\ &= U (\lambda I) c \end{aligned}$$

$$w_k = U (\lambda_1 \dots \lambda_n)^k c$$

where $\lambda I = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$
 $\lambda_1 > \dots > \lambda_n$
as eigenvalue
of Σ

$$W_t = U \begin{pmatrix} \lambda_1 & & \\ & \dots & \\ & & \lambda_n \end{pmatrix}^t c \quad \text{when } t \rightarrow \infty \quad W_t = U \lambda_1^t c$$

$$\begin{bmatrix} \bigcirc \\ \triangle \\ \square \\ \times \end{bmatrix}^T = \begin{bmatrix} \text{wavy} \\ \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ & \ddots \\ 0 & \lambda_n \end{bmatrix} \begin{bmatrix} \bigcirc \\ \triangle \\ \square \\ \times \end{bmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax+by & ay+bw \\ cx+dz & cy+dw \end{pmatrix}$$

$$\begin{bmatrix} \text{wavy} \\ \text{---} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} u_{11}\lambda_1, \lambda_{12}\lambda_2, \dots \end{bmatrix}$$

$$\begin{bmatrix} \text{---} \end{bmatrix} \begin{bmatrix} \bigcirc \\ \triangle \\ \square \\ \times \end{bmatrix} = 0$$

let $U_i c = z_0(i)$
 $W_t(i) = \lambda_i^t z_0(i)$

Then focus on dynamics of U_1

$$W_t(i) = U_i \lambda_i^t c = \lambda_i^t U_i c$$