17A_Q3 - f

固有ベクトルの直行行列

Let $U = [v_1 \cdots v_p]$ … denote orthogonal matrix of eigenvectors of Σ

$$\Sigma = U \Gamma V^T \quad \text{(Singular Value Decomposition)}$$

$$A = X \Lambda X^{-1} \quad \text{(Eigenvalue Decomposition)}$$

$$U \Sigma U^{-1} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

scalar dynamics,

$$w_{t+1} = \Sigma w_t \qquad z_t(i) = \lambda_i^t z_0(i)$$

$$z_0(i) = \langle v_i, w_0 \rangle \quad \text{where}$$

Σの固有ベクトルv_iを縦ベクトルとして並べた行列をPとすると$P^{-1}\Sigma P$は固有値を対角成分に持つ対角行列となる

縦ベクトル

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_{11}b_1 + a_{12}b_2 \\ a_{21}b_1 + a_{22}b_2 \end{pmatrix}$$

縦ベクトル (column vector)

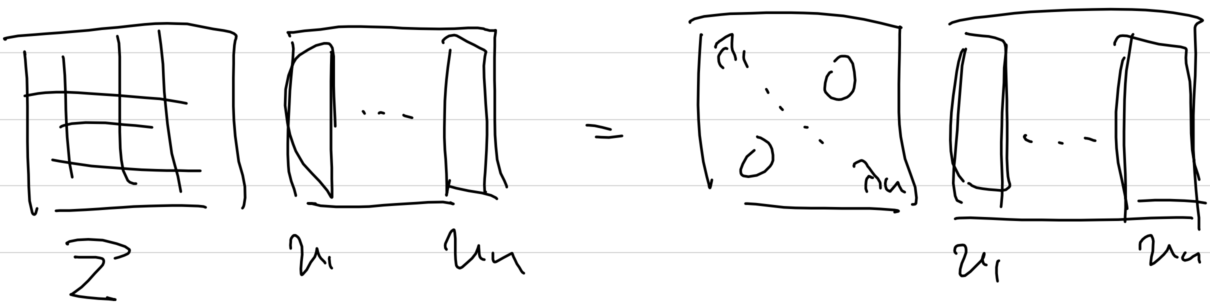
横ベクトル

$$\begin{pmatrix} a_1 & a_2 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_1b_{11} + a_2b_{21}, & a_1b_{12} + a_2b_{22} \end{pmatrix}$$

$$U = [u_1, \dots u_n]$$

$$U^{-1}\Sigma U = \Lambda \iff \Sigma U = \Lambda U$$

$$\Sigma u_i = \lambda_i u_i$$

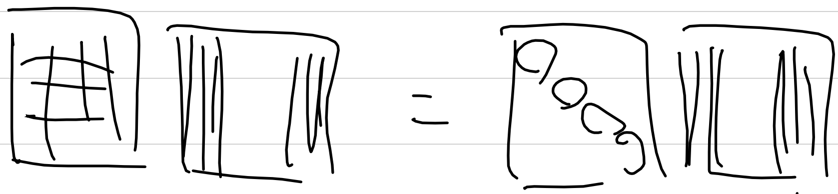


①

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix}$$

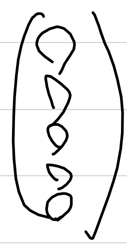
②

$$\Sigma V = \Lambda V$$

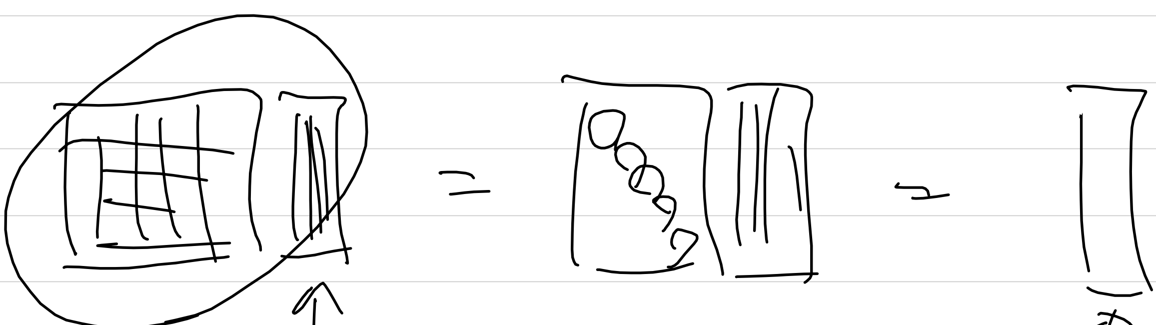


を縦ベクトルごとに考えると

$V = [v_1, \dots, v_n]$ に相当する

$v_i =$  みたいな感じ

$$\Sigma v_i = \Lambda v_i$$



このベクトルを x とする

i-th eigenvector

$\langle v_i, v_0 \rangle = z_0(i)$

$x^{(0)} = c_1 u_1 + \cdots \quad c_n u_n$ とする

線形独立な固有ベクトル

$= Uc$

$x^{(0)}$ に Σ をかけたものを $x^{(1)}$ とすると

$$x^{(1)} = \quad x^{(0)} = \Sigma U c = U U^{-1} \Sigma U c$$

$$= U \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} c$$

$$= \lambda_1 c_1 u_1 + \cdots \quad \lambda_n c_n u_n$$

k回 Σ をかけることをくり返す

$$x^{(k)} = \underbrace{\Sigma \cdots \Sigma}_{k\text{個}} U c = \lambda_1^k c_1 u_1 + \cdots \lambda_n^k c_n u_n$$

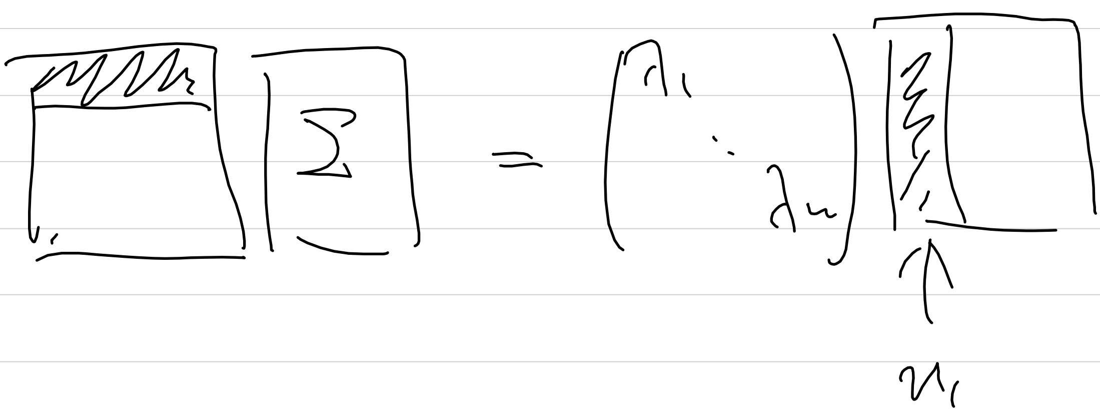
$$= \lambda_1^k c_1 \left\{ u_1 + \cdots \underbrace{\left(\frac{\lambda_n}{\lambda_1}\right)^k}_{\downarrow\, 0} \left(\frac{c_n}{c_1}\right) u_n \right\}$$

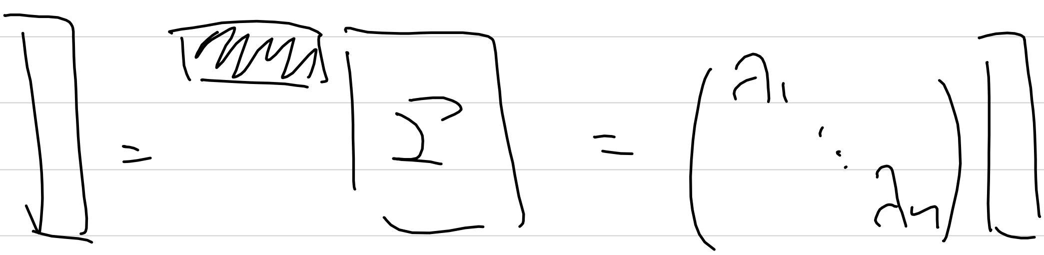
$$\lim_{k \to \infty} x^{(k)} = \lambda_1^k c_1 u_1 = u_1$$

定数倍も固有ベクトル

$U=[u_1 \cdots u_n]$ 直行ベクトルで Σ の固有ベクトル

$$U\Sigma U^{-1} = \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix} \Leftrightarrow U\Sigma = \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix} U$$





$z_0(i) = \langle u_i, w_0 \rangle$

$w_{t+1} = \Sigma w_t$

行列Σが対角化可能とすると
線形独立なベクトルuを使って あるベクトルx

$$x_0 = c_1u_1 + \cdots c_nu_n$$
$$= Uc$$

は上記のように表せる

$$x_1 = \Sigma x_0 = \Sigma U c = UU^{-1}\Sigma U c$$
$$= U\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}c$$
$$= \lambda_1 c_1 u_1 + \cdots \lambda_n c_n u_n$$

$\Leftarrow$

$$x_{t+1} = \Sigma x_t$$
$$= \lambda_1^t c_1 u_1 + \cdots \lambda_n^t c_n u_n$$ と表せる

ここで あるベクトル x を u とすると

$$u_{t+1} = \Sigma u_t$$
$$= \lambda_1^{(t+1)} c_1 \left\{ u_1 + \underbrace{\left(\frac{\lambda_2}{\lambda_1}\right)^{(t+1)}}_{\downarrow\, 0} \frac{c_2}{c_1} u_2 + \cdots \underbrace{\left(\frac{\lambda_n}{\lambda_1}\right)^{t+1}}_{\downarrow\, 0} \frac{c_n}{c_1} u_n \right\}$$

$n \to \infty$

$$u_{t+1} = \lambda_1^{(t+1)} c_1 u_1 \longrightarrow$$

定数倍も固有ベクトル

$$\left(\text{e.g.} \quad x_2 = \Sigma x_1 = UU^{-1} \Sigma x_1 = UU^{-1} \Sigma U (\lambda \cdots \lambda_n) c = U (\lambda \cdots \lambda_n)^2 c \right)$$

17A_Q3-f Answer

Let $w \in \mathbb{R}^p$ as an arbitrary vector

w_0 can be expressed by using

sequence of scalar c and

independent vectors $U = [u_1 \cdots u_n]$

as follows

$$w_0 = c_1 u_1 + \cdots c_n u_n = \underline{\underline{Uc}}$$

$$w_1 = \Sigma w_0 = \Sigma \underline{\underline{Uc}}$$

$$\left(\begin{array}{l} \text{then } U \text{ is eigenvectors of } \Sigma \\ \text{in other word} \quad \Sigma U = U \lambda I \\ \qquad\qquad \Leftrightarrow \quad U^{-1} \Sigma U = \lambda I \end{array} \right)$$

$$w_1 = \Sigma Uc = UU^{-1} \Sigma_1 Uc$$
$$= U(\lambda I)c$$

where $\lambda I = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

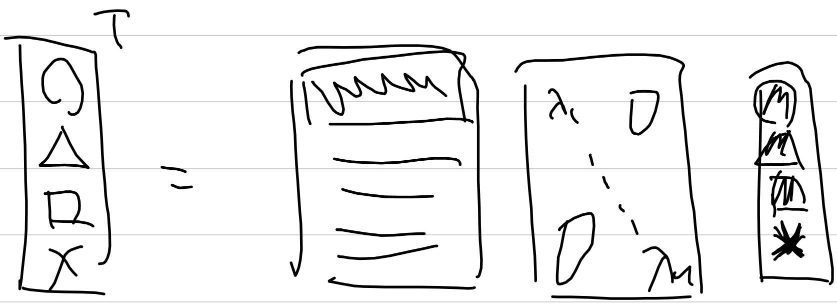
$\lambda_1 > \cdots \lambda_n$

as eigenvalue

of Σ

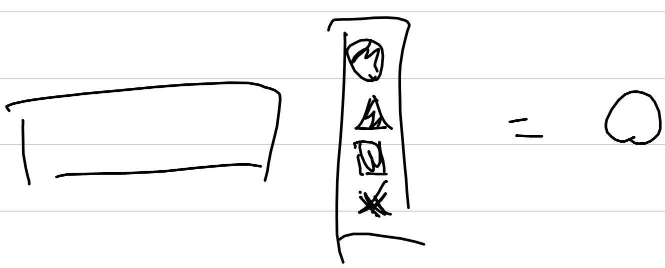
$$w_t = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^n c$$

$w_t = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^t c$ when $t \to \infty$ $w_t = u \lambda_1^t c$



$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax + bz, & ay + bw \\ \cancel{c \cdots} & \end{pmatrix}$

$u_{11}\lambda_1, \lambda_{12}\lambda_2, \ldots$



let $u_i c = z_0(i)$

$w_t(i) = \lambda_i^t z_0(i)$

Then focus on dynamics of u_1

$w_t(i) = u_i \lambda^t(i) c = \lambda^t(i) u_i c$