17AQ3-7

let

$$w_0 = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \qquad v = \begin{pmatrix} v_1 \\ \vdots \\ v_p \end{pmatrix}$$

$$\begin{aligned}
\mathbb{E}[\langle w_0, v\rangle^2] &= \mathbb{E}\left[\left(\sum_{i=1}^{p} x_i v_i\right)^2\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{p}\sum_{j=1}^{p} x_i x_j v_i v_j\right] \\
&= \sum_{i=1}^{p}\sum_{j=1}^{p} \mathbb{E}[x_i x_j v_i v_j] \\
&= \sum_{i=1}^{p}\sum_{j=1}^{p} v_i v_j \, \mathbb{E}[x_i x_j]
\end{aligned}$$

↑ v is unit-length vector

$$w_0 = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \sim N\left(0, \frac{1}{p} I_{p\times p}\right)$$

$$\mathrm{Cov}(x_i x_j) = \underbrace{\mathbb{E}[x_i x_j]}_{\begin{cases} 0 & \text{if } i \neq j \\ \frac{1}{p} & \text{if } i = j \end{cases}} - \underbrace{\mathbb{E}[x_i]}_{=0}\underbrace{\mathbb{E}[x_j]}_{=0}$$

from definition

covariance

variance (diagonal)

element of variance-covariance-matrix

Then,

$$\mathbb{E}[\langle w_0 , u\rangle^2] = \sum_{i=1}^{p} \sum_{j=1}^{p} u_i u_j \, \mathbb{E}[x_i x_j]$$

$$= \underbrace{\sum_{i=1}^{p} \|u_i\|^2}_{=1} \times \underbrace{\sum_{i=1}^{p} \sum_{j=1}^{p} \mathbb{E}[x_i x_j]}_{\frac{1}{p}}$$

since unit vector

$$= \frac{1}{p}$$

17A_Q3 - f

固有ベクトルの直行行列

Let $U = [v_1 \cdots v_p]$ ... denote

orthogonal matrix of eigenvectors of Σ

$$\left(\begin{array}{ll} \Sigma = U \Gamma V^T & \text{(Singular Value Decomposition)} \\ A = X \Lambda X^{-1} & \text{(Eigenvalue Decomposition)} \end{array} \right)$$

$$U \Sigma U^{-1} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

scalar dynamics,

$$w_{t+1} = \Sigma w_t \qquad z_t(i) = \lambda_i^t z_0(i)$$

$$z_0(i) = \langle v_i, w_0 \rangle$$ where

Σの固有ベクトルv_iを縦ベクトルとして並べた行列をPとすると $P^{-1}\Sigma P$は固有値を対角成分に持つ対角行列となる

縦ベクトル

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \underbrace{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}} = \begin{pmatrix} a_{11}b_1 + a_{12}b_2 \\ a_{21}b_1 + a_{22}b_2 \end{pmatrix}$$

縦ベクトル (column vector)

横ベクトル

$$\begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_1 b_{11} + a_2 b_{21}, & a_1 b_{12} + a_2 b_{22} \end{pmatrix}$$

$$U = [u_1, \cdots u_n]$$

$$U^{-1} \Sigma U = \Lambda \iff \Sigma U = \Lambda U$$

$$\Sigma u_i = \lambda_i u_i$$

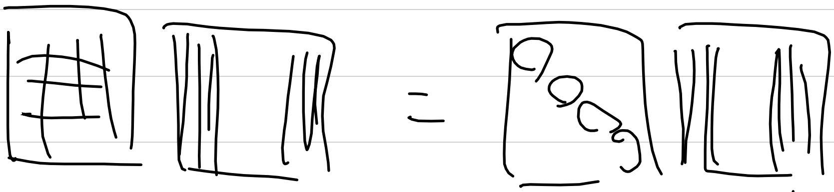
$$\underset{\Sigma}{\Sigma} \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}$$

①

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix}$$

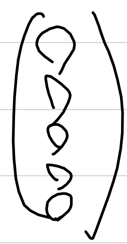
②

$\Sigma V = \Lambda V$

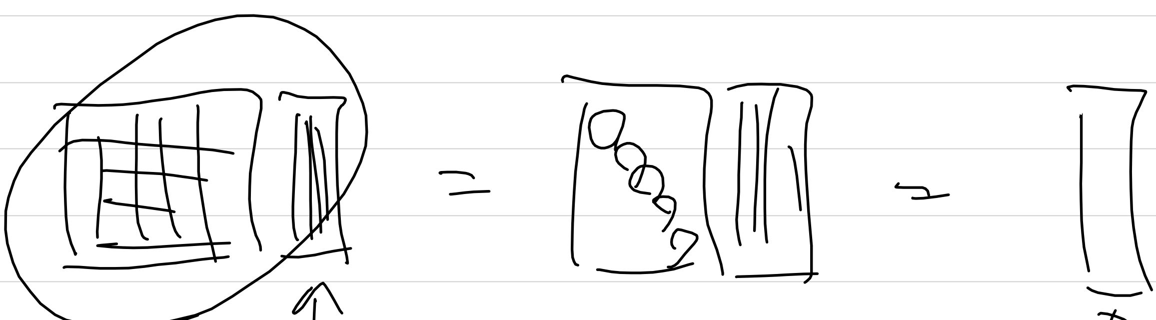


を縦ベクトルごとに考えると

$V = [v_1, \ldots, v_n]$ に相当する

$v_i =$  みたいな感じ

$\Sigma v_i = \Lambda v_i$



このベクトルを x とする

i-th eigenvector

$\langle v_i, w_0 \rangle = z_0(i)$

$x^{(0)} = c_1 u_1 + \cdots \quad c_n u_n$ とする

線形独立な固有ベクトル

$= Uc$

$x^{(0)}$ に Σ をかけたものを $x^{(1)}$ とすると

$$x^{(1)} = \quad x^{(0)} = \Sigma Uc = \underline{UU^{-1}\Sigma U} c$$

$$= U \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} c$$

$$= \lambda_1 c_1 u_1 + \cdots \quad \lambda_n c_n u_n$$

k回 Σ をかけることをくり返す

$$x^{(k)} = \underbrace{\Sigma \cdots \Sigma}_{k個} Uc = \lambda_1^k c_1 u_1 + \cdots \lambda_n^k c_n u_n$$

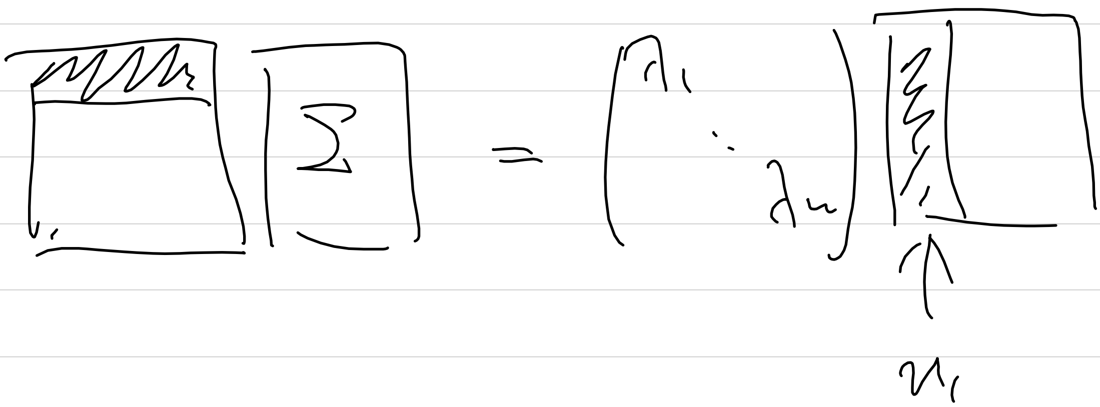
$$= \lambda_1^k c_1 \left\{ u_1 + \cdots \underbrace{\left(\frac{\lambda_n}{\lambda_1}\right)^k}_{\downarrow\, 0} \left(\frac{c_n}{c_1}\right) u_n \right\}$$

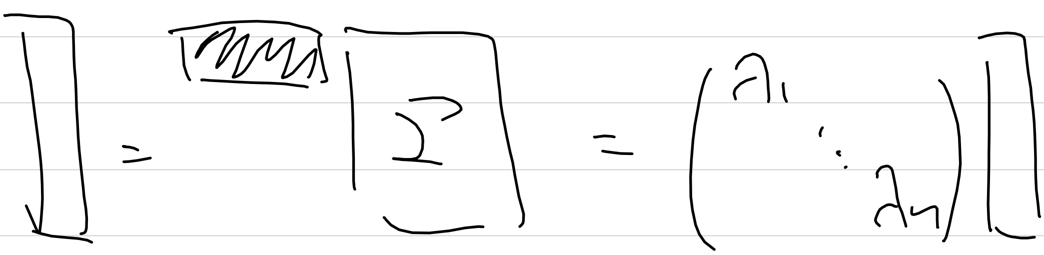
$$\lim_{k \to \infty} x^{(k)} = \lambda_1^k c_1 u_1 = v_1$$

定数倍も固有ベクトル

$U = [u_1 \cdots u_n]$ 直行ベクトルで Σ の固有ベクトル

$$U \Sigma U^{-1} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \Leftrightarrow U \Sigma = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U$$





$x_0(i) = \langle u_i, w_0 \rangle$

$w_{t+1} = \Sigma w_t$

行列 Σ が対角化可能とすると
線形独立なベクトル u を使って あるベクトル x

$$x_0 = c_1 u_1 + \cdots c_n u_n$$
$$= Uc$$

は上記のように表せる

$$x_1 = \Sigma x_0 = \Sigma U c = U U^{-1} \Sigma U c$$

$$= U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} c$$

$$= \lambda_1 c_1 u_1 + \cdots \lambda_n c_n u_n$$

$\Leftarrow$

$$x_{t+1} = \Sigma x_t$$
$$= \lambda_1^t c_1 u_1 + \cdots \lambda_n^t c_n u_n$$ と表せる

ここで あるベクトル x を u とする

$$w_{t+1} = \Sigma w_t$$
$$= \lambda_1^{(t+1)} c_1 \left\{ u_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{(t+1)} \frac{c_2}{c_1} u_2 + \cdots \left(\frac{\lambda_n}{\lambda_1}\right)^{t+1} \frac{c_n}{c_1} u_n \right\}$$

$$\left(\frac{\lambda_2}{\lambda_1}\right)^{(t+1)} \to 0, \quad \left(\frac{\lambda_n}{\lambda_1}\right)^{t+1} \to 0$$

$n \to \infty$

$$w_{t+1} = \lambda_1^{(t+1)} c_1 u_1 \longrightarrow$$

定数倍も固有ベクトル

$$\left(\begin{aligned} \text{e.g.} \quad x_2 = \Sigma x_1 &= UU^{-1} \Sigma x_1 \\ &= UU^{-1} \Sigma U \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} c \\ &= U \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^2 c \end{aligned} \right)$$