The Power Method

arbitrary initialization $w_0 \in \mathbb{R}^p$

Σ : empirical covariance matrix

< Standard Eigenvalue Problem >

$$Ax = \lambda x \qquad x \neq 0$$

$$\Leftrightarrow (A - \lambda I)x = 0$$

λ : eigen value
x : eigen vector

Algorithm of the power method

Let $x^{(0)}$ as $\|x^{(0)}\|_2 = 1$ as initial vector
$k = 0$

Update $x^{(k+1)}$ as follows

$$x^{(k)} \leftarrow \frac{x^k}{|x^k|} \quad \text{(normalization)}$$

$$x^{(k+1)} \leftarrow Ax^{(k)}$$

$$\lambda^{(k)} = x^{(k)} x^{(k+1)} \quad \text{(Rayleigh Quotient)}$$

$\uparrow$

cuz. $\lambda^{(k)} = \dfrac{x^{(k)} x^{(k+1)}}{x^{(k)} x^{(k)}}$ $\because (x^{(k)})^2 = 1$

$$\lim_{k \to \infty} \frac{x^{kT} x^{k}}{x^{k} x^{k-1}} = \lambda_1$$

x^k が 絶対値最大の 固有値 λ_1 に属する
固有ベクトル方向に
漸近していく

べき乗法の 気持ち

$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ みたいになると

扱いやすい $\Rightarrow$ 対角化したい

行列 A が対角化可能 $\Rightarrow$ 線形独立な
固有ベクトルを用いて

あるベクトル $x^{(0)}$ を 任意の定数ベクトル
$C = (c_1 \cdots c_n)$
を使って表現可

$x^{(0)} = c_1 u_1 \cdots c_n u_n$

$= Pc$

$$x^{(1)} = Ax^{(0)}$$

$$= APc$$

$$= IAPC$$

$$= \underbrace{PP^{-1}AP}_{\lambda\text{の対角化}}C = P\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}c$$

$$= \lambda_1 c_1 u_1 \cdots \lambda_n c_n u_n$$

$$x^{(k)} = \lambda_1^k c_1 u_1 \cdots \lambda_n^k c_n u_n$$

Aが対角可能 $\Rightarrow |\lambda_1| > |\lambda_2| > \cdots |\lambda_n|$

より

$$x^{(k)} = \lambda^k c_1 \left\{ u_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k \frac{c_2}{c_1} u_2 \cdots \left(\frac{\lambda_n}{\lambda_1}\right)^k \frac{c_n}{c_1} u_n \right\}$$

$|\lambda_1| > |\lambda_2|$ より $\lim_{k\to\infty} \left|\frac{\lambda_2}{\lambda_1}\right| = 0$

$\Downarrow$

$$\lim_{k\to\infty} x^{(k)} = \lambda_1^k c_1 u_1$$

固有ベクトル u_1 を定数倍しても固有ベクトルなので

$$= u_1$$

ここで Power Method

適当な初期値 $x^{(0)}$ から

$$x^{(k)} = A x^{(k-1)} \quad \text{により}$$

x を更新していくと絶対値 max の A の

固有ベクトル v_1 が得られる

$|x^{(k)}| = 1$ となるように正規化は要因する

Rayleigh Quotient

$$R(v_1) = \frac{v_1 A v_1}{v_1 v_1} = \frac{v_1 \lambda v_1}{v_1 v_1} = \frac{v_1 \lambda v_1}{|v_1|^2} = \lambda_1$$

↓

$$R(x^{(k)}) = \frac{x^{(k)} A x^{(k)}}{x^{(k)} x^{(k)}} = \frac{x^{(k)} x^{(k+1)}}{x^{(k)} x^{(k)}}$$

↓

$$\lambda^{(k)}$$

収束判定は

$$|A x^{(k)} - \lambda^{(k)} x^{(k)}| < \varepsilon$$

1. $x^{(0)}$ as initialization, $k=0$

2. $x^k \leftarrow \frac{x^k}{|x^k|}$

3. $x^{(k+1)} = Ax^{(k)}$

4. $\lambda^{(k)} = x^{(k)} x^{(k+1)}$

5. if $||x^{(k+1)}|^2 - |\lambda^{(k)}|^2| < \varepsilon^2$ なら終了

else

Square 2-Norm

$$||x||_2 := \sqrt{|x_1|^2 + \cdots |x_n|^2}$$

$X_1 \cdots X_n$ が $\sim N(0,1)$

$\Rightarrow Y = X_1^2 + \cdots X_n^2$ は $\chi^2(n)$

自由度nのカイ二乗分布

Theorem

自由度 n のカイ二乗分布に従う variable Y の

$E(X) = n$

Chi-Square Distribution

Theorem

X が $N(0,1)$ に従うとき

$Y = X^2$ は 自由度1のカイ二乗分布に従う.

$$X \sim N(0,1) \Rightarrow X^2 \sim \chi^2(1)$$

$\chi^2(n)$ の PDF

$$p(x) = \begin{cases} \dfrac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} & x > 0 \\ 0 & (x < 0) \end{cases}$$

によって表される分布

Let PDF of Gaussian $X \sim N(0, 1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp -\left(\frac{x^2}{2}\right)$$

$Y = X^2$ PDF which follow is denoted as $g(y)$

$$g(y) = \frac{d}{dy} P(X^2 \le y)$$

↳ CDF (cumulative distribution function)

↑

PDF is differential of

$$g(y) = \frac{d}{dy} P(-\sqrt{y} < x < \sqrt{y})$$

$$= \frac{d}{dy} \int_{-\sqrt{y}}^{\sqrt{y}} f(x)\, dx$$

F as primitive function

$$g(y) = \frac{d}{dy}\int_{-\sqrt{y}}^{\sqrt{y}} f(x)\,dx$$

$F'(\sqrt{y}) = (\sqrt{y})' f(\sqrt{y})$ if F is primary function of f

$$g(y) = \underbrace{\frac{d}{dy}F(\sqrt{y})}_{\hookrightarrow (\sqrt{y})' f(\sqrt{y}) = F'(\sqrt{y})} - \frac{d}{dy}F(-\sqrt{y})$$

$$= F'(y) - F'(-y)$$

$$= (\sqrt{y})' f(\sqrt{y}) - (-\sqrt{y})' f(-\sqrt{y})$$

$$= \frac{1}{2}y^{-\frac{1}{2}} f(\sqrt{y}) + \left(\frac{1}{2}y^{-\frac{1}{2}}\right) f(-\sqrt{y})$$

$$= \frac{1}{2\sqrt{y}}\left(f(\sqrt{y}) + f(-\sqrt{y})\right)$$

$$= \frac{1}{2\sqrt{y}} \times \left(\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y}{2}\right) + \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y}{2}\right)\right)$$

$$= \frac{1}{\cancel{2}\sqrt{y}} \times \cancel{2} \times \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y}{2}\right)$$

$$= \frac{1}{2^{\frac{1}{2}}\sqrt{\pi}}\, y^{-\frac{1}{2}} \exp\left(-\frac{y}{2}\right)$$

$$g(y) = \frac{1}{2^{\frac{1}{2}}\sqrt{\pi}} y^{-\frac{1}{2}} \exp\left(-\frac{y}{2}\right)$$

$$= \frac{1}{2^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)} y^{-\frac{1}{2}} \exp\left(-\frac{y}{2}\right)$$

Chi - Square Distribution

$$g(x) = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} \quad (x > 0)$$

then $g(y)$ = Chi - Square Distribution

when $n = 1$

$E[g(y)] = 1$ because $n = \underline{1}$

自由度 n の Chi-Square Distribution

$$X \sim \chi^2(n)$$

$$\mathbb{E}[X] = n$$

Proof

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\, p(x)\, dx$$

$$= \frac{1}{2^{\frac{2}{n}} \Gamma\left(\frac{n}{2}\right)} \int_0^{\infty} x^{\frac{n}{2}} e^{-\frac{x}{2}}\, dx$$

$t = \frac{x}{2}$ とすると

$$\frac{1}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^{\infty} (2t)^{\frac{n}{2}} e^{-t}\, 2\, dt$$

$$= \frac{2}{\Gamma\left(\frac{n}{2}\right)} \int_0^{\infty} e^{-t} t^{\frac{n}{2}}\, dt$$

Gamma Function Definition

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx \quad \text{and} \quad \underline{\Gamma(s+1) = s\Gamma(s)}$$

Then

$$\int_0^\infty e^{-t} t^{\frac{n}{2}}\, dt = \int_0^\infty e^{-t} t^{(\frac{n}{2}+1)-1}\, dt$$

$$= \Gamma\left(\frac{n}{2}+1\right)$$

Therefore

$$E[X] = \frac{2}{\Gamma(\frac{n}{2})} \Gamma\left(\frac{n}{2}+1\right)$$

$$= \frac{2}{\Gamma(\frac{n}{2})}\left(\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\right)$$

$$= 2 \times \frac{n}{2} = \underline{n}$$

To answer question of

expected square-2 norm of w_0

where $w_0 \sim N(0, \frac{1}{p} I_{p \times p})$

$w_0 \in \mathbb{R}^p$

1st, Definition of square-2norm of matrix A

is as follows

$$\|A\|_2 = \sqrt{|a_1|^2 + \cdots |a_n|^2}$$

2nd assume $w_0 \sim N(0, \frac{1}{p} I_{p \times p})$

to follow gaussian distribution

value of $|a_1|^2 + \cdots |a_n|^2 = Y$

follows Chi-Square Distribution

3 Expected value of Y (which follows Chi-Square Distribution) is

$$E(Y) = 1$$

$$(x_1)^2 + \cdots (x_n)^2 = Y$$

自由度 n のカイ二乗分布

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$w_0 \sim N\left(0, \frac{1}{p} I_{p \times p}\right) \sigma$$

$(w_1)^2 + \cdots (w_p)^2 = Y$ とすると.

$$g(y) = \frac{1}{2\sqrt{y}} \left(f(\sqrt{y}) + f(-\sqrt{y}) \right)$$

$$= \frac{1}{2\sqrt{y}} \times \left\{ \frac{1}{\sqrt{2\pi}\sqrt{p}} \exp\left(-\frac{y}{2p^{-1}}\right) + \frac{1}{\sqrt{2\pi}p} \exp\left(-\frac{y}{2p^{-1}}\right) \right\}$$

$$= \frac{1}{\sqrt{y}} \frac{1}{\sqrt{2\pi}\sqrt{p^{-1}}} \exp\left(-\frac{y}{2p^{-1}}\right) \qquad \hat{y} = yp$$

$$g(\hat{y}) = \frac{1}{2^{\frac{1}{2}} \Gamma(\frac{1}{2})} \hat{y}^{-\frac{1}{2}} \times \exp\left(-\frac{\hat{y}}{2}\right)$$

$g(\hat{y})$ は自由度 P の

カイ二乗分布

$$\mathbb{E}[\hat{y}] = P$$

$\hat{y} = yP$

$y = \frac{\hat{y}}{P}$ より

$$\mathbb{E}[y] = 1$$

$\sigma_i = 1$ より $w_0 \sim N(0, \frac{1}{P} I_{P\times P})$

$w_0 \in \mathbb{R}^P$

の期待値は 1 となる。